On finite-time stabilization of evolution equation: Homogeneous approach

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Homogeneity

Dilation in the Banach space $\mathbf{B}$ and its generator

Definition

A map $\mathbf{d} : \mathbb{R} \to \mathcal{L}(\mathbf{B}, \mathbf{B})$ is called **dilation** in $\mathbf{B}$ if it satisfies

- **group property:** $\mathbf{d}(0) = I$, $\mathbf{d}(t+s) = \mathbf{d}(t)\mathbf{d}(s)$, $t, s \in \mathbb{R}$;
- **strong continuity:** the map $\mathbf{d}(\cdot)u : \mathbb{R} \to \mathbf{B}$ is continuous, $u \in \mathbf{B}$;
- **limit property:** $\lim\limits_{s \to -\infty} \|\mathbf{d}(s)u\| = 0$ and $\lim\limits_{s \to +\infty} \|\mathbf{d}(s)u\| = \infty$ uniformly on $\mathcal{S} = \{u \in \mathbf{B} : \|u\| = 1\}$.

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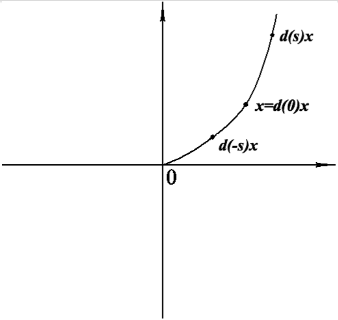

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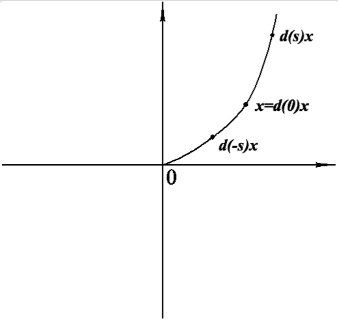
Linear operator $G_{\mathbf{d}} : \mathcal{D}(G_{\mathbf{d}}) \subset \mathbf{B} \to \mathbf{B}$

$$G_{\mathbf{d}} u = \lim_{s\to 0} s^{-1}(\mathbf{d}(s)u - u) \quad (1)$$

is **generator** of **d**



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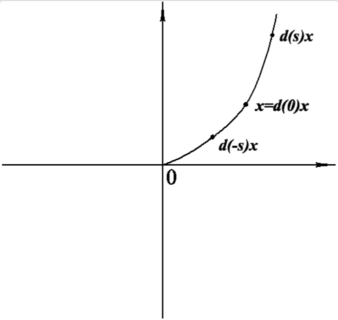

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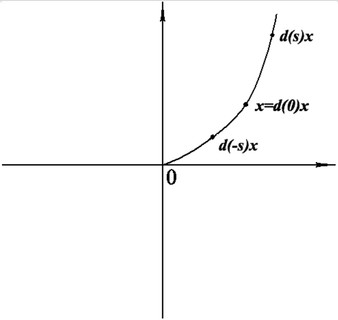
$$\mathcal{D}(G_{\mathbf{d}})=\{u\in\mathbf{B}:\exists\lim_{s\to 0}s^{-1}(\mathbf{d}(s)u-u)\}$$

dense in $\mathbf{B}$ and if $u\in\mathcal{D}(G_{\mathbf{d}})$ one has

$$\frac{d}{ds}\mathbf{d}(s)u=G_{\mathbf{d}}\mathbf{d}(s)u=\mathbf{d}(s)G_{\mathbf{d}}u.$$



Examples of dilations

Uniform(standard) dilation

$$\mathbf{d}(s) = e^s, \quad G_{\mathbf{d}} = I \in \mathcal{L}(\mathbf{B}, \mathbf{B})$$

Weighted dilation

$$\mathbf{B} = \mathbb{R}^n, \quad \mathbf{d}(s) = \mathrm{diag}\{e^{r_i s}\} \in \mathbb{R}^{n\times n}, \quad r_i > 0, \quad G_{\mathbf{d}} = \mathrm{diag}\{r_i\} \in \mathbb{R}^{n\times n}$$

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Example (Dilation in L^2)

$\mathbf{B} = L^2(\mathbb{R}, \mathbb{R}), \qquad (\mathbf{d}(s)u)(x) = e^s u(e^{\mu s}x), \qquad s \in \mathbb{R}, u \in \mathbf{B}, x \in \mathbb{R}$

$G_{\mathbf{d}} u = \mu x u' + u$ with $\mathcal{D}(G_{\mathbf{d}}) = \{u \in L^2 : xu' \in L^2\}$.

Monotone dilation

Definition

The dilation $\mathbf{d}$ is said to be **monotone** on **B** if:

$$\|\mathbf{d}(s)\|_{\mathcal{L}} := \sup_{u\in\mathcal{S}} \|\mathbf{d}(s)u\| < 1 \quad \text{if} \quad \forall s < 0. \tag{2}$$

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Example (Monotonicity is depended on norm in **B**)

$B = \mathbb{R}^n$ with $\|u\| = \sqrt{u^\top P u}$, $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$ is monotone **if and only if**

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Proposition

If the dilation **d** *is monotone on* **B** *then* $\lfloor \mathbf{d}(s) \rfloor_{\mathcal{L}} := \inf_{u\in\mathcal{S}} \|\mathbf{d}(s)u\| > 1$, $s > 0$;

- $\forall u \in$ **B** *there exists* **unique** *pair* $(s_0, u_0) \in \mathbb{R} \times \mathcal{S} \,:\, u = \mathbf{d}(s_0)u_0$;
- $\|\mathbf{d}(\cdot)u\| : \mathbb{R} \to \mathbb{R}_+$ *is strictly monotone increasing if* $u \neq 0$.

Canonical Homogeneous Norm

Definition (Homogeneous norm)

A functional $p : \mathbf{B} \to \mathbb{R}_+$ is said to be **d**-homogeneous norm on $\mathbf{B}$ if $p(u) \to 0$ as $u \to \mathbf{0}$ and $p(\mathbf{d}(s)u) = e^s p(u) > 0$ for $u \in \mathbf{B}\backslash\{\mathbf{0}\}$, $s \in \mathbb{R}$.

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Lemma (On Canonical Homogeneous Norm in Banach Space)

Let **d** be monotone on **B** and $\exists \beta > 0$: $\|\mathbf{d}(s)\|_{\mathcal{L}} \leq e^{\beta s}$ for $s < 0$ then $\|\cdot\|_{\mathbf{d}}$ is Lipschitz continuous on $\mathbf{B}\backslash\{\mathbf{0}\}$.

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Lemma (On Canonical Homogeneous Norm in Hilbert Space)

Let $\mathbf{d}$ be monotone on $\mathbf{H}$ and $\exists\beta>0$:$\langle G_{\mathbf{d}}u,u\rangle\ge\beta\|u\|^2$, $u\in\mathcal{D}(G_{\mathbf{d}})$, then $\|\cdot\|_{\mathbf{d}}$ is Fréchet differentiable on $\mathcal{D}(G_{\mathbf{d}})\backslash\{\mathbf{0}\}$

$$\left(D\|u\|_{\mathbf{d}}\right)(\cdot)=\frac{\langle\mathbf{d}(-\ln\|u\|_{\mathbf{d}})\cdot,\mathbf{d}(-\ln\|u\|_{\mathbf{d}})u\rangle}{\langle G_{\mathbf{d}}\mathbf{d}(-\ln\|u\|_{\mathbf{d}})u,\mathbf{d}(-\ln\|u\|_{\mathbf{d}})u\rangle}\|u\|_{\mathbf{d}}. \tag{4}$$

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An operator $f:\mathcal{D}\to\mathbf{B}$ is $\mathbf{d}$-homogeneous of degree ν if $\mathbf{d}(s)\mathcal{D}\subset\mathcal{D}$ and

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Example

Operator : $A=\frac{\partial^2}{\partial x^2}:D(A)\subset L^2(\mathbb{R},\mathbb{R})\to L^2(\mathbb{R},\mathbb{R})$ and $D(A)=H^2(\mathbb{R},\mathbb{R})$

Dilation : $(\mathbf{d}(s)u)(x)=e^{s}u(e^{-\mu s}x), \qquad s\in\mathbb{R}, u\in L^2, x\in\mathbb{R}$

$$G_{\mathbf{d}}u=-\mu xu'+u \quad \text{with} \quad \mathcal{D}(G_{\mathbf{d}})=\{u\in L^2 : xu'\in L^2\}$$

Homogeneity : $\mathbf{d}(s)D(A)\subset D(A)$ and

$$A(\mathbf{d}(s)u)(x)=\frac{\partial^2 e^{s}u(e^{-\mu s}x)}{\partial x^2}=e^{-2\mu s}e^{s}\left.\frac{\partial^2 u(y)}{\partial y^2}\right|_{y=e^{-\mu s}x}=e^{-2\mu s}\mathbf{d}(s)(Au)(x)$$

Finite-time stabilization of evolution systems

Finite-time stabilization problem

Let us consider the following linear equation

$$\frac{d}{dt}u(t) = Au(t) + B\xi(u(t)), \quad t > 0 \tag{6}$$

$$u(0) = u_0 \in \mathcal{D}(A), \tag{7}$$

where

- the operators $A : \mathcal{D}(A) \to \mathbf{H}$ and $B : \mathcal{D}(B) \subset \mathbf{B} \to \mathbf{H}$ are linear,
- $u(t) \in \mathcal{D}(A) \subset \mathbf{H}$ is the system state,
- $\xi : \mathcal{D}(A) \subset \mathbf{H} \to \mathcal{D}(B) \subset \mathbf{B}$ is a bounded (locally or globally) feedback control to be designed in order **to stabilize the zero solution of the system in a finite time**.

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C) $\exists \tilde{A} : \mathcal{D}(\tilde{A}) \subset \mathbf{H} \to \mathbf{H}$ – $\mathbf{d}$-homogeneous extension[a] of A with the **negative** degree $-\mu$ and $\exists K \in \mathcal{L}(\mathbf{H}, \mathbf{B})$: $BK(D(A)) \subset D(A)$ and

$$\beta\|z\|^2 \leq \alpha \left\langle G_{\mathbf{d}} z, z \right\rangle \leq -\langle z, (\tilde{A} + BK)z\rangle, \qquad z \in \mathcal{D}(\tilde{A}),$$

where $\alpha > 0, \beta > 0$ and $G_{\mathbf{d}}$ is the generator of $\mathbf{d}$, $\mathcal{D}(\tilde{A}) \subset \mathcal{D}(G_{\mathbf{d}})$;

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D) $\exists \nu \geq \mu - \omega^- \, : \, e^{-\nu s}\mathbf{d}(s)BKz = BK\mathbf{d}_2(s)z, \quad \forall z \in \mathcal{D}(A),$

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then

- the feedback (8) is

$$\xi(u) = \begin{cases} \|u\|_{\mathbf{d}}^{\nu-\mu} K \mathbf{d}_1(-\ln(\|u\|_{\mathbf{d}}))u & \text{if} \quad u \neq 0, \\ 0 & \text{if} \quad u = 0 \end{cases} \tag{8}$$

- locally bounded on $\mathbf{H}\backslash\{\mathbf{0}\}$, Fréchet differentiable on $\mathcal{D}(G_{\mathbf{d}})\backslash\{\mathbf{0}\}$,
- continuous at $\mathbf{0} \in \mathbf{H}$ provided that $\nu + \omega^- > \mu$,
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  - globally bounded $(\|\xi\| \leq M\|K\|_{\mathcal{L}})$ on $\mathbf{H}$ provided that $\nu + \omega^+ = \mu$;

- the closed-loop system has a unique classical solution[a] such that

  $$u(t) \to \mathbf{0} \quad \text{as} \quad t \to T(u_0)$$

  with the setting time $T(u_0) \leq \frac{\|u_0\|_{\mathbf{d}}^{\mu}}{\alpha\mu}$.

---

[a] A continuous function $u : [0, T) \to \mathcal{D}(A)$ is said to be classical solution to the initial value problem (6), (7) if it is continuously differentiable on $(0, T)$, $u(0) = u_0$ and $\frac{d}{dt}u(t) = Au(t) + B\xi(u(t))$ for all $t \in (0, T)$.

Proof of the main theorem(the key ideas)

I. Existence and uniqueness of the classical solution

$$\frac{d}{dt}u_g(t) = (A + R(t))u_g(t), \quad u_g(0) = u_0 \in \mathcal{D}(A)$$

$$R(t) = Bg^{\nu-\mu}(t)K\mathbf{d}_1(-\ln g(t)) \in \mathcal{L}(\mathbf{H},\mathbf{H}), \quad g \in C((0,T),\mathbb{R}_+).$$

If $g(t) = \|u_g(t)\|_{\mathbf{d}}$ then $u_g \in C^1((0,T),\mathcal{D}(\tilde{A}))$ is a solution to (6).

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II. Finite-time convergence

$$\tfrac{d}{dt}\|u(t)\|_{\mathbf{d}} = (D_u\|u(t)\|_{\mathbf{d}})(\dot{u}(t)) = \frac{e^{s(t)}\langle \mathbf{d}(-s(t))\dot{u}(t), \mathbf{d}(-s(t))u(t)\rangle}{\langle G_{\mathbf{d}}\mathbf{d}(-s(t))u(t), \mathbf{d}(-s(t))u(t)\rangle},$$

and

$$\mathbf{d}(-s(t))\dot{u}(t) = \mathbf{d}(-s(t))(Au(t) + B\xi(u(t))) = \frac{(\tilde{A}+BK)\mathbf{d}(-s(t))u(t)}{e^{-\mu s(t)}},$$

imply

$$\tfrac{d}{dt}\|u(t)\|_{\mathbf{d}} \leq -\alpha e^{(1-\mu)s(t)} = -\alpha\|u(t)\|_{\mathbf{d}}^{1-\mu}.$$

Some remarks

Remark (On Set-Valued Extension)

If $\tilde{A} : D(\tilde{A}) \subset \mathbf{H} \rightrightarrows \mathbf{H}$ is the set-valued extension of A, i.e. $Au \subset \tilde{A}u$ for all $u \in \mathcal{D}(A) \subset \mathcal{D}(\tilde{A})$, then Main theorem remains true provided that

$$\beta\|z\|^2 \leq \alpha \langle G_{\mathbf{d}} z, z\rangle \leq -\langle z, y + BKz\rangle, \qquad z \in \mathcal{D}(\tilde{A}) \text{ and } y \in \tilde{A}z,$$

replaces Condition C.

Remark (On piecewise linear feedback)

If

$$\xi_{sw}(u) = \begin{cases} r_i^{\nu-\mu} K \mathbf{d}_1(-\ln(r_i))u & \text{if } \|u\|_{\mathbf{d}} \in (r_{i+1}, r_i], \\ 0 & \text{if } u = 0 \end{cases} \qquad r_i > 0 \qquad (9)$$

then Main Theorem remains true for properly selected $r_i > 0$.

Examples

Example 1: Stabilization of the double integrator

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$$\xi(u) = K\mathbf{d}(-\ln \|u\|_{\mathbf{d}})u$$ - the stabilizing feedback

Example 2: Distributed stabilization of heat system on $[0, 1]$

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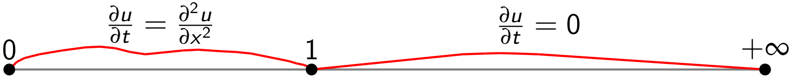
Step1 Construct an operator $A : \mathbf{H} \to \mathbf{H} = L^2(\mathbb{R}_+, \mathbb{R})$ that is connection of the Laplace operator with homogeneous Dirichlet boundary conditions on $[0,1]$ and the zero operator on $[1,+\infty)$:



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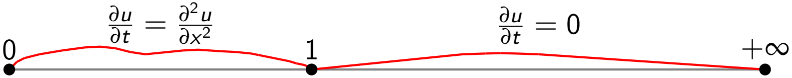

Step2 Select the dilation $\mathbf{d}(s)=\mathbf{d}_1(s)\mathbf{d}_2(s)$ on $\mathbf{H}$, where

$$\mathbf{d}_1(s)=e^{\nu s},\quad (\mathbf{d}_2(s)u)(x)=u(e^{-\mu s}x) \quad \text{and}$$

$$\mathcal{D}(G_{\mathbf{d}})=\{u\in\mathbf{H}: xu'\in L^2\}$$

and design a homogeneous extension $\tilde{A}:\mathbf{H}\rightrightarrows\mathbf{H}$ of the operator A

$$\tilde{A}\mathbf{d}(s)u=e^{-2\mu s}\mathbf{d}(s)\tilde{A}u$$

$$Au\subset\tilde{A}u \quad u\in\mathcal{D}(A)\subset\mathcal{D}(\tilde{A})\subset\mathcal{D}(G_{\mathbf{d}})$$

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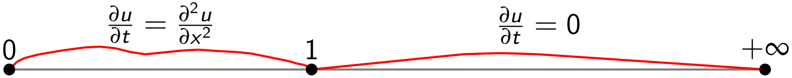

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Step3 Apply the Main Theorem to $\dot{u}(t) = Au(t) + \xi(u(t)), \quad t > 0$

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- Let us consider the **operator**

$$Au = \left\{ \begin{array}{ccc} \frac{\partial^2 u}{\partial x^2} & \text{if} & x \in (0,1) \\ 0 & \text{if} & x \geq 1 \end{array} \right\}$$

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- The domain of A is

$$\mathcal{D}(A) = \left\{ z \in \mathbf{H} : \substack{z_{[0,1]} \in H_0^1((0,1),\mathbb{R}) \cap H^2((0,1),\mathbb{R}), \\ xz' \in \mathbf{H} \text{ and } xz^2(x) \to 0 \text{ as } x \to +\infty} \right\} \subset \mathcal{D}(G_{\mathbf{d}}),$$

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- A is the generator of the strongly continuous semigroup $\{\Phi\}_{t \geq 0}$ on $\mathbf{H}$

$$(\Phi(t)u)(x) = \left\{ \begin{array}{ll} \int_0^1 M(t,x,y)u(y)dy & \text{if } x \in (0,1), \\ u(x) & \text{if } x \in (1,+\infty), \end{array} \right.$$

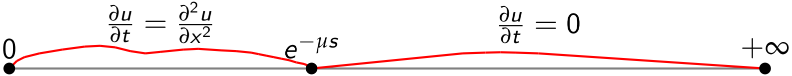
  where $M(t,x,y)$ is the heat kernel on $[0,1]$.

Step 2 : Construction of homogeneous extension of A

- Bu analogy with A let us introduce the family of operators ($s \in \mathbb{R}$)

$$A_s u = \begin{cases} \frac{\partial^2 u}{\partial x^2} & \text{if} \quad x \in (0, e^{-\mu s}), \\ 0 & \text{if} \quad x \geq e^{-\mu s}, \end{cases}$$

with the domain $\mathcal{D}(A_s) = \mathbf{d}(s)\mathcal{D}(A)$. Obviously, $A = A_0$.



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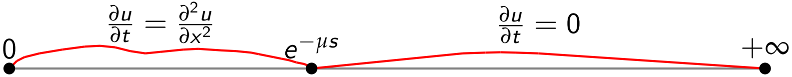

- The set-valued homogeneous extension $\tilde{A} : \mathcal{D}(\tilde{A}) \subset \mathbf{H} \to \mathbf{H}$ is

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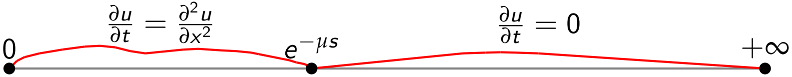

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- Since $A_{s+q}\mathbf{d}(s)u = e^{-2\mu s}\mathbf{d}(s)A_q u$ for $u \in \mathcal{D}(A_q)$, $s, q \in \mathbb{R}$ then

$$\tilde{A}\mathbf{d}(s)u = e^{-2\mu s}\mathbf{d}(s)\tilde{A}u \quad \text{for} \quad u \in \mathcal{D}(\tilde{A}).$$

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Let $B = I \in \mathcal{L}(\mathbf{H}, \mathbf{H})$, $K = -\rho I$ where $\rho \geq \alpha(\nu + 0.5\mu)$ and $\alpha > 0$.

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  a) $\langle A_s z, z\rangle = -0.5\langle z', z'\rangle$;
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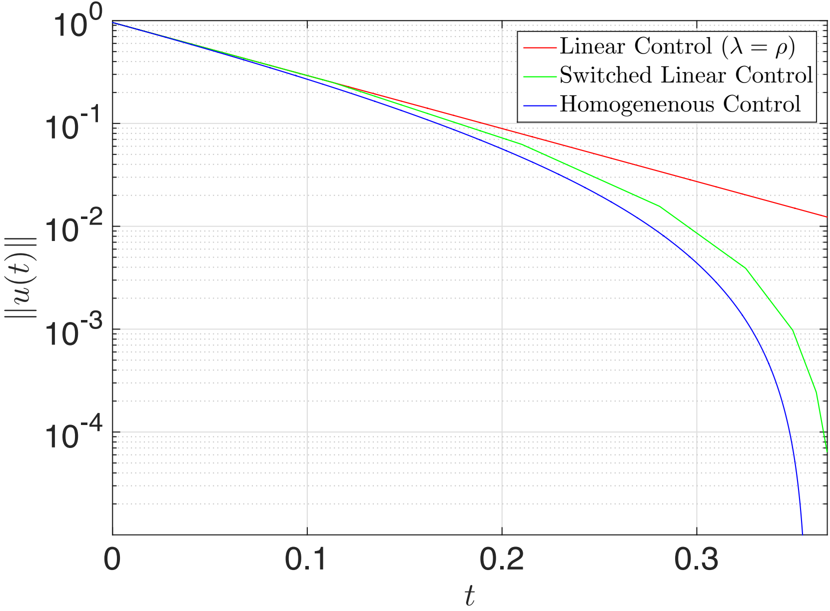
- **The closed-loop system:**

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \rho\frac{u}{\|u\|^{\mu/(\nu+0.5\mu)}}, \qquad \begin{array}{c} u(t,0)=u(t,1)=0, \\ u(0,x)=u_0(x),\ x\in[0,1]. \end{array}$$

Example 2: Simulation results



Summary

Publications

- Polyakov, Coron, Rosier, CDC 2016
- IEEE TAC (ready to be submitted)

Future works

- PDEs on *n*-dimensional domains
- Time delay systems
- Non-linear PDEs (e.g. Burgers equation)

- Homogeneous observer design
- Generalizations of **d**-homogeneity