Finite-time boundary stabilization of heat system using a switched state feedback control

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Problem statement

Model Description

Let us consider the heat system

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \quad v(0) = v_0 \tag{1}$$

with boundary control ζ

$$(v(t))(0) = 0 \qquad (v(t))(1) = \zeta(t), t > 0 \tag{2}$$

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The control aim is to steer the state $v(t)$ of the system (1), (2) *to zero in a finite time* by means of a linear switching feedback control

$$\zeta(t) = K_{\sigma(t)} v(t), \tag{3}$$

where $K_i : L^2((0,1), \mathbb{R}) \to \mathbb{R}$, $i \in \mathbb{Z}$ and $\sigma : \mathbb{R}_+ \to \mathbb{Z}$

$$\sigma(t) = G(\sigma(t^-), v(t^-)), \tag{4}$$

with $G : \mathbb{Z} \times L^2((0,1), \mathbb{R}) \to \mathbb{Z}$, $t^- = t + 0^-$ and $\sigma(0) \in \mathbb{Z}$.

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- $\sigma(t)$ satisfies the discrete equation (4) for all $t \in (0,T)$
- v is a solution to the heat system (1) - (3) understood in **weak sense**:

$$-\int_0^1 v_0(x)\xi(0,x)dx - \int_0^T\int_0^1 v(t,x)\xi_t(t,x)dxdt$$

$$+\int_0^T K_{\sigma(t)} v(t,\cdot)\xi_x(t,1)dt - \int_0^T\int_0^1 v(t,x)\xi_{xx}(t,x)dxdt = 0$$

for all $\xi \in C^2([0,T)\times[0,1],\mathbb{R})$ with compact support in $[0,T)\times[0,1]$ such that ξ vanishes at $[0,T)\times\{0,1\}$.

The basic ideas

Stabilization by means of switched linear feedback

$$\dot{x}(t) = u(t), \quad t > 0, \tag{5}$$

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$$u_{pl}(x(t)) = -\frac{x(t)}{r_i} \quad \text{if} \quad |x(t)| \in (r_{i+1}, r_i], \quad i \in \mathbb{Z}, \tag{6}$$

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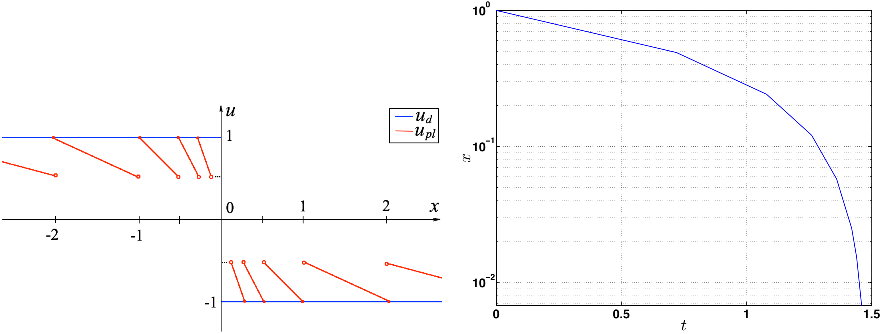

Figure : Relay feedback, its approximation ($r_i = 2^{-i}$) and simulation

The scheme needs modification in high dimensional case

$$\begin{aligned}\dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= u(t)\end{aligned} \qquad x(t) = (x_1(t), x_2(t))^\top \in \mathbb{R}^2 \tag{7}$$

$$u_{sw}(t) = \frac{1}{r_i}(k_1 x_1(t) + k_2 x_2(t)) \quad \text{if} \quad \|x\|_{\mathbb{R}^2} \in (r_{i+1}, r_i]. \tag{8}$$

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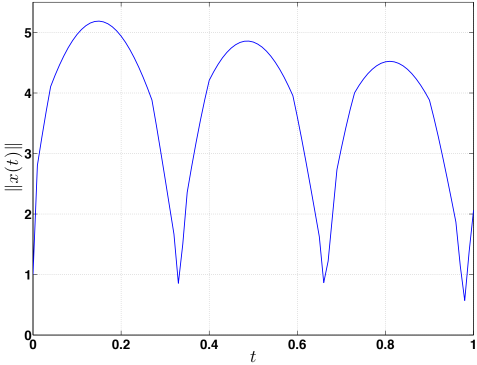


Figure : Solution to system (7) with piecewise linear control (8)

Idea 1 : Switched control with state depended switchings

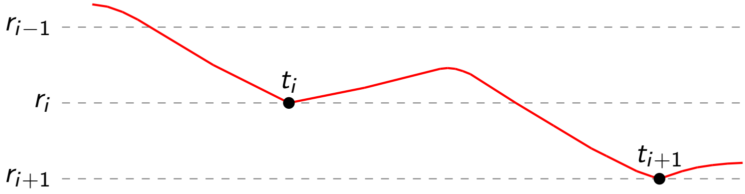
$$\dot{z}(t) = Az + Bu(t)$$

The linear switched control is

$$u(t) = K_i z(t) \quad \text{if} \quad \sigma(t) = i \tag{9}$$

with the switching law

$$\sigma(t) = \sigma(t_i^-) + 1 \quad \text{if} \quad \|z(t_i)\| = r_i$$



Idea 2 : Backstepping transformation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \quad v(0) = v_0$$
$$(v(t))(0) = 0 \qquad (v(t))(1) = \zeta(t),$$

$$\zeta(t) = \int_0^1 k(1, y, \lambda) v(t, y) dy \quad \text{with} \quad k(x, y, \lambda) = -\frac{\lambda y I_1(\sqrt{\lambda(x^2 - y^2)})}{\sqrt{\lambda(x^2 - y^2)}} \quad (10)$$

and I_m with $m \in \mathbb{Z}$ is the modified Bessel function of the first kind.

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$$(F(\lambda)u)(x) = -\int_0^x k(x, y, \lambda) u(y) dy \quad (11)$$

The **state transformation** $u = v + F(\lambda)v$ applied to (1), (2) yields

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \lambda u, \qquad u(t, 0) = 0, \quad u(t, 1) = 0. \quad (12)$$

The *inverse transformation* is $v = u + F(-\lambda)u$, where $u, v \in L^2((0, 1), \mathbb{R})$.

Control design

Estimates for the norms of transfomration and its inverse

Proposition (1)

If $z \in L^2((0,1),\mathbb{R})$ *then*

$$\|z+F(\lambda)z\| \leq \Psi_1(\lambda)\|z\| \text{ and } \|z+F(-\lambda)z\| \leq \Psi_{-1}(\lambda)\|z\|,$$

$$\Psi_1(\lambda) = 1+\lambda\frac{\sqrt{\int_0^1\int_0^x y^2\left(I_0\left(\sqrt{\lambda(x^2-y^2)}\right)-I_2\left(\sqrt{\lambda(x^2-y^2)}\right)\right)^2 dydx}}{2},$$

$$\Psi_{-1}(\lambda) = 1+\lambda\frac{\sqrt{\int_0^1\int_0^x y^2\left(J_0\left(\sqrt{\lambda(x^2-y^2)}\right)+J_2\left(\sqrt{\lambda(x^2-y^2)}\right)\right)^2 dydx}}{2},$$

where J_k *is the Bessel function of the first kind.*

Switched Linear Control

$$\xi(t) = \int_0^1 k\left(1, y, 2^{\sigma(t)}\right) v(t,y)dy, \qquad \sigma(t) = G(\sigma(t^-), v(t^-)), \quad (13)$$

$$G(\sigma, v) = \begin{cases} i+1 & \text{if} \quad \sigma = i \text{ and } \|v\| \leq r_{i+1}, \\ i & \text{if} \quad \sigma = i \text{ and } r_{i+1} < \|v\| < r_{i-1}, \\ i-1 & \text{if} \quad \sigma = i \text{ and } \|v\| \geq r_{i-1}, \end{cases} \quad (14)$$

where $r_0 = 1$, $r_i = e^{-q_i} r_{i-1}$, $i \in \mathbb{Z}$ and the numbers q_i are defined by (15).

Proposition (2)

If

$$q_i = \ln \Psi_1\left(2^i\right) + \ln \Psi_{-1}\left(2^i\right), \quad (15)$$

then $q_i > 0$ *for all* $i \in \mathbb{Z}$, $\qquad r_i \to 0$ *as* $i \to +\infty$ $\qquad$ *and*

$$\lim_{i \to +\infty} \frac{q_{i+1}}{q_i} = \sqrt{2}.$$

Main Theorem

For any initial condition

$$v(0) = v_0 \in L^2((0,1), \mathbb{R}) \tag{16}$$

and

$$\sigma(0) = i_0 \in \mathbb{Z} \quad \textit{such that} \quad \|v(0)\| \in (r_{i_0+1}, r_{i_0}], \tag{17}$$

the closed-loop system (1), (2), (13), (4), (14) *has a unique solution* $(\sigma, v) : C((0,T), \mathbb{R} \times L^2((0,1), \mathbb{R})$ *such that*

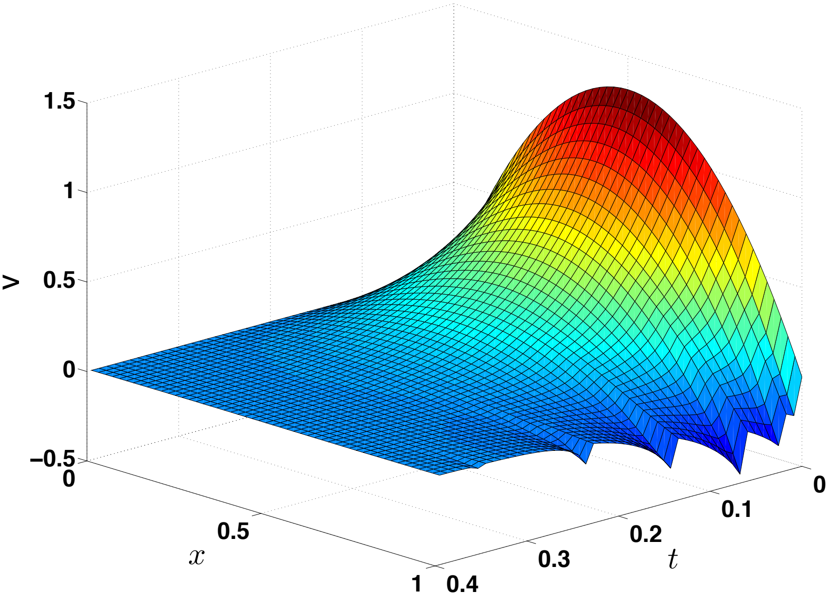
$$\|v(t)\| \to 0 \quad \textit{as} \quad t \to T^-,$$

where

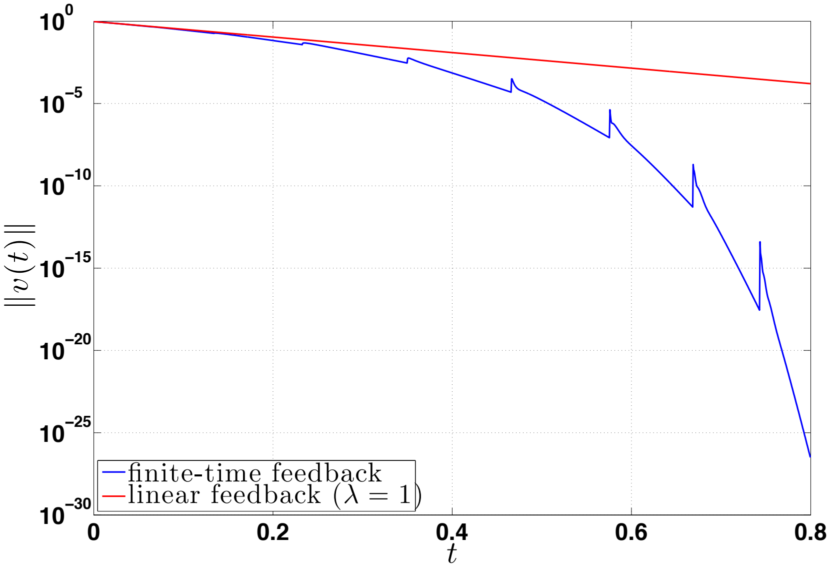
$$T \leq \sum_{i=i_0}^{+\infty} \frac{q_i + q_{i+1}}{2^i} < +\infty. \tag{18}$$

Example

Example (Simulation results)



Example (Comparison with linear feedback)



Thank you very much for your attention