Finite time Stability

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Examples

Let $\alpha \in]0,1[$, then the system

$$\dot{x} = -\lceil x \rfloor^{\alpha},\ x \in \mathbb{R}, \tag{1}$$

($\lceil x \rfloor = \mathrm{sign}(x)x$) for any initial condition $x_0 \in \mathbb{R}$ and $t \geq 0$ has the solution:

$$\Phi^t(x_0) = \begin{cases} s(t,x_0) & \text{if} \quad 0 \leq t \leq T(x_0) \\ 0 & \text{if} \quad t > T(x_0) \end{cases}, \tag{2}$$

with $s(\tau,x) = \mathrm{sign}(x)\left(|x|^{1-\alpha} - \tau(1-\alpha)\right)^{\frac{1}{1-\alpha}}$ and $T(x) = \frac{|x|^{1-\alpha}}{1-\alpha}$, thus the solutions reach the origin in the finite time $T(x_0)$.

Examples

For this example we can notice that:

- Finite Time Stability (FTS) "looks like" an infinite eigenvalue assignation for the closed-loop system at the origin, therefore the right-hand side of the ordinary differential equation *cannot be locally Lipschitz at the origin*;
- there exists the *settling time function* $T(x_0)$ that determines the time for a solution to reach the equilibrium, this function depends on the initial condition of a solution.

The main issue with T is its continuity at the origin. For continuous systems, the continuity of T at 0 is equivalent to the continuity of T everywhere

Examples

The bi-limit homogeneity application allows us to have a globally bounded T (in some special cases), which means that in practice one gets a fixed time of convergence to the origin for all initial conditions. For an example consider a simple system

$$\dot{x} = -\lceil x \rfloor^{1/2} - \lceil x \rfloor^{3/2},\ x \in \mathbb{R}, \tag{3}$$

which for any $x_0 \in \mathbb{R}$ and $t \geq 0$ has the solution

$$\Phi^t(x_0) = \begin{cases} v(t, x_0) & \text{if} \quad 0 \leq t \leq T(x_0) \\ 0 & \text{if} \quad t > T(x_0) \end{cases}, \tag{4}$$

where $v(\tau, x) = \tan[\arctan(\sqrt{|x|}) - 0.5\tau]^2 \mathrm{sign}(x)$ and the settling time function $T(x) = 2\arctan(\sqrt{|x|})$. As we can conclude from these expressions, if $|x| \to +\infty$, then $T(x) \to \pi$, therefore all solutions approach the origin in a time less than π.

Definitions

$$\dot{x} = f(x, d), \quad t \in \mathbb{R}_+, \tag{5}$$

where x is the state which belongs to an open set $\mathcal{X} \subset \mathbb{R}^n$ containing the origin, d is the time-dependent disturbance (for example, a Lebesgue measurable signal $d : \mathbb{R}_+ \to \mathbb{R}^p$).

Definitions

Definition

The system (5) for $d = 0$ is said to be **finite-time stable (FTS)** at the origin (on an open neighbourhood $\mathcal{V} \subset \mathbb{R}^n$) if:

1. there exists a function $T : \mathcal{V} \setminus \{0\} \to \mathbb{R}_+$ such that for all $x_0 \in \mathcal{V} \setminus \{0\}$, $\Phi^t(x_0)$ is defined (and unique) on $[0, T(x_0))$ and $\lim_{t \to T(x_0)} \Phi^t(x_0) = 0$. T is called the *settling-time function* of the system (5).
2. there exists a function $\delta \in \mathcal{K}$ such that for all $x_0 \in \mathcal{V}$ we have $\|\Phi^t(x_0)\| \leq \delta(\|x_0\|)$ for all $t \geq 0$.

If $\mathcal{V} = \mathbb{R}^n$, then the system is called globally FTS. Furthermore, if the property 1) is fulfilled only, then the origin of the system (5) is said to be finite-time attractive.

Definitions

It is possible to show that if the system (5) is FTS, then it is also asymptotically stable with a continuous flow for all $x_0 \in \mathcal{V}$ and it has the uniqueness and completeness in forward time for all solutions initiated in $\mathcal{V}$ [1]. In particular, at the origin the system has a unique solution $x(t,0) = 0$ for all $t \geq 0$, thus we can extend the definition above taking $T(0) = 0$.

Definitions

Definition

The system (5) for $d = 0$ is said to be **FxTS** at the origin if it is globally FTS and the settling-time function T is bounded, i.e. $T(x) \leq T_{\max}$ for some $T_{\max} > 0$ and for all $x \in \mathbb{R}^n$.

Results

Let $x \in \mathbb{R}$ and consider the simple scalar equation $\dot{x} = -r(x)$ if solution reach zero then the time to reach the origin is

$$T = \int_0^{x_0} \frac{dx}{r(x)}$$

Lemma

[13] The origin of $\dot{x} = -r(x), x \in \mathbb{R}$ *is GFTS iff* $xr(x) > 0, \forall x \in \mathbb{R} \setminus \{0\}$ *and* $\int_0^{x_0} \frac{dx}{r(x)} < \infty, \forall x_0 \in \mathbb{R}$.

Results

Let us consider

$$\dot{x}_1 = -\lceil x_1 \rfloor^{\frac{1}{2}} + 0.5 \lceil x_2 \rfloor^{\frac{1}{2}} \quad (6)$$

$$\dot{x}_2 = 0.5 \lceil x_1 \rfloor^{\frac{1}{2}} - \lceil x_2 \rfloor^{\frac{1}{2}} \quad (7)$$

Take $V(x) = 0.5(x_1^2 + x_2^2)$ then

$$\dot{V} = -\left(|x_1|^{\frac{3}{2}} + |x_2|^{\frac{3}{2}} \right) + 0.5(x_1 \lceil x_2 \rfloor^{\frac{1}{2}} + x_2 \lceil x_1 \rfloor^{\frac{1}{2}}) \quad (8)$$

(Symmetry if $x_1 \rightleftarrows x_2$).
Consider

$$S = \{x \in \mathbb{R}^2 : |x_1|^{\frac{3}{2}} + |x_2|^{\frac{3}{2}} = 1\}$$

Clearly $\dot{V}$ has max when $x_1 = x_2$ on S, and in that case
$\dot{V} = -0.5 \left(|x_1|^{\frac{3}{2}} + |x_2|^{\frac{3}{2}} \right) = -0.5$
thus $\dot{V}$ is negative (homogeneity)

Results

Let $h(\dot{V}, V) = \dot{V} + V^{\frac{4}{5}} = h(x)$. h is max on S when $x_1 = x_2$ and the max is

$$h(x) = -0.5 + |x_1|^{\frac{8}{5}} < 0.$$

But on S we have

$$|x_1| = 2^{-\frac{2}{3}}.$$

Thus we conclude that

$$\dot{V} \leq -V^{\frac{4}{5}} \tag{9}$$

(here $r(V) = V^{\frac{4}{5}}$) **GFTS**.

Results

In order to have sufficient and equivalent Lyapunov characterizations for ODE (5) with $d = 0$:

Definition

A class $\mathcal{K}$ function r belongs to *class $\mathcal{KI}$* if $r \in \mathcal{CL}$ and there exists $\epsilon > 0$ such that:

$$\int_0^{\epsilon} \frac{dz}{r(z)} < +\infty.$$

Results

Let $V : \mathcal{V} \to \mathbb{R}_+$ be a Lyapunov function (it is continuously differentiable, positive definite and radially unbounded) and r be from the class $\mathcal{KI}$, then the first condition is that for all $x \in \mathcal{V}$:

$$\dot{V}(x) \leq -r[V(x)]. \tag{10}$$

The existence of such a pair (V, r) is also a necessary condition for FTS in some particular cases:

- scalar case (see before),
- systems with forward uniqueness of solutions with continuous settling time function at the origin (see below),
- homogeneous systems.

Results

Let $\mathcal{E}$ and $\mathcal{F}$ be two vector spaces, one denotes by $\mathcal{L}^0(\mathcal{E},\mathcal{F})$ (respectively $\mathcal{L}^k(\mathcal{E},\mathcal{F})$) the set of locally Lipschitz (respectively C^k) function on $\mathcal{E}\setminus\{0\}$ with value in $\mathcal{F}$ and $\mathcal{CL}^0(\mathcal{E},\mathcal{F})$ (respectively $\mathcal{CL}^k(\mathcal{E},\mathcal{F})$) the set of continuous function on $\mathcal{E}$, locally Lipschitz (respectively C^k) on $\mathcal{E}\setminus\{0\}$ with value in $\mathcal{F}$.

Results

A particular form of (10) is described as follow:

$$\dot{V}(x) \leq -aV(x)^{\alpha}, \; a > 0, \; \alpha \in (0,1). \tag{11}$$

In this case $r(s) = as^{\alpha}$ and it is obviously from the class $\mathcal{KI}$.

Theorem

[2] Consider the system (5) for $d = 0$ with the forward uniqueness of solutions outside the origin, then the following properties are equivalent:

- (i) *the origin of the system (5) is* **FTS** *with a continuous settling-time function T,*
- (ii) *there is a Lyapunov function V satisfying the condition (11),*
- (iii) *there is a Lyapunov function V and a class $\mathcal{KI}$ function r satisfying the condition (10).*

Moreover, if V is a Lyapunov function satisfying the condition (11), then

$$T(x) \leq \frac{V(x)^{1-\alpha}}{c(1-\alpha)}.$$

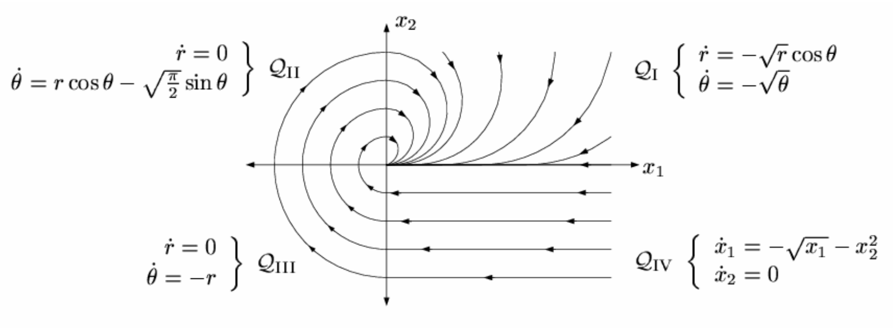
Results

Bhat et al. gives an example where T is not continuous at the origin. $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined on the quadrants

$$\begin{aligned}
\mathcal{Q}_{\mathsf{I}} &= \left\{x \in \mathbb{R}^2 \setminus \{0\} : x_1 \geq 0, x_2 \geq 0\right\}, \\
\mathcal{Q}_{\mathsf{II}} &= \left\{x \in \mathbb{R}^2 \setminus \{0\} : x_1 < 0, x_2 \geq 0\right\} \\
\mathcal{Q}_{\mathsf{III}} &= \left\{x \in \mathbb{R}^2 \setminus \{0\} : x_1 \leq 0, x_2 < 0\right\}, \\
\mathcal{Q}_{\mathsf{IV}} &= \left\{x \in \mathbb{R}^2 \setminus \{0\} : x_1 > 0, x_2 < 0\right\}.
\end{aligned}$$

as shown in the figure below with $f(0) = 0$, $r > 0$, $\theta \in [0, 2\pi[$ and $x = (x_1, x_2) = (r\ \cos(\theta), r\ \sin(\theta))$.

Results



x_2
x_1
$\dot{r} = 0$
$\dot{\theta} = r\cos\theta - \sqrt{\frac{\pi}{2}}\sin\theta$
$\mathcal{Q}_{\mathrm{II}}$
$\mathcal{Q}_{\mathrm{I}}$
$\dot{r} = -\sqrt{r}\cos\theta$
$\dot{\theta} = -\sqrt{\theta}$
$\dot{r} = 0$
$\dot{\theta} = -r$
$\mathcal{Q}_{\mathrm{III}}$
$\mathcal{Q}_{\mathrm{IV}}$
$\dot{x}_1 = -\sqrt{x_1} - x_2^2$
$\dot{x}_2 = 0$

Results

It follows from [5, Proposition 2.2] and [4, Ch 10, lemma 2] that the system defined on previous figure possesses unique solution in forward time on the negative x_2-axis $\mathcal{X}_2^-$. With the stability of the system, it is not enough to ensure that the settling time function is defined on $\mathcal{X}_2^-$. So, the system can not be finite time stable in the sense of definition **??** because the settling time is not well defined. So, we have to use the following condition for the system (5) for $d = 0$:

Condition

The right way to study finite time stability involving the existence of a settling time function as in definition 1 is that the system (5) for $d = 0$ possesses uniqueness of solutions outside the origin.

Results

For DI

$$\dot{x} \in F(x, d), \quad t \in \mathbb{R}_+, \tag{12}$$

for which a sufficient stability condition can be formulated only. In this case the condition (11) has to be rewritten as follows

$$\sup_{\varphi \in F(x,0)} DV(x)\varphi \leq -r[V(x)], \tag{13}$$

where as before $r \in \mathcal{KI}$, thus we do not assume anymore that V is continuously differentiable.

Theorem

[7] Let $0 \in F(0,0)$ *in the system (12). If there exists a Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *(locally Lipschitz continuous, positive definite and radially unbounded) that verifies the condition (18), then the origin is FTS for (12) with* $d = 0$*. Moreover, the settling-time function* T *of (12) satisfies:*

$$T(x) \leq \int_0^{V(x)} \frac{dz}{r(z)}.$$

Results

For the FxTS property the following sufficient Lyapunov conditions have been proposed in [8]:

Theorem

Let there exist a Lipschitz continuous, positive definite and radially unbounded Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that for all* $x \in \mathbb{R}^n$*:*

$$\sup_{\varphi \in F(x,0)} DV(x)\varphi \leq -[\alpha V^p(x) + \beta V^q(x)]^k,$$

for some strictly positive α, β, p, q, k *with* $pk < 1$ *and* $qk > 1$, *then the system (12) with* $d = 0$ *is FxTS and*

$$T(x) \leq \frac{1}{\alpha^k(1-pk)} + \frac{1}{\beta^k(qk-1)}, \;\forall x \in \mathbb{R}^n. \qquad (14)$$

Results

As we can conclude from this result, due to the property $pk < 1$, locally around zero the function V is an FTS Lyapunov function for (12). The time estimate (14) follows from an analogous estimate of Theorem 6 under substitution in the integral limits $V(x) = +\infty$ for $r(z) = [\alpha z^p + \beta z^q]^k$.

Results

Now let us pass to the case where $d \neq 0$ and analyse the influence of d on the FTS property (robustness).

Definition

The system (12) is called finite-time integral input-to-state stable (finite-time iISS) if for all $x_0 \in \mathbb{R}^n$ and $d(t) \in \mathcal{L}_\infty$ the estimate

$$\alpha(\|\Phi^t(x_0)\|) \leq \beta(\|x_0\|, t) + \int_0^t \gamma(\|d(\tau)\|)d\tau$$

is satisfied for all $t \geq 0$ and $\Phi^t(x_0) \in \mathbb{S}(x_0)$ for some $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{GKL}$ and $\gamma \in \mathcal{K}$.

The UFTS property means that the system stability is not influenced by the inputs d from $\mathcal{D}$. The finite-time ISS and finite-time iISS quantify the deviations of trajectories for bounded and integrally bounded inputs respectively.

Results

The Lyapunov characterization of uniform UFTS coincides with the one introduced for FTS before, the only additional requirement is that the condition (18) is satisfied for all $d \in \mathcal{D}$. To introduce the Lyapunov functions for finite-time ISS and finite-time iISS we will need the following property: for two functions $a_1, a_2 : \mathbb{R}^n \to \mathbb{R}_+$ the relation $a_1(x) \succ a_2(x)$ means that there exists $\epsilon > 0$ such that $a_1(x) \geq a_2(x)$ for all $\|x\| \leq \epsilon$.

Results

Definition

A locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called finite-time ISS Lyapunov function for the system (12) if there are $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^p$

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|),$$

$$\sup_{\varphi \in F(x,d)} DV(x)\varphi \le -\alpha_3(\|x\|) + \sigma(\|d\|), \tag{15}$$

with $\alpha_3(\|x\|) \succ aV^\alpha(x)$ for some $a > 0$ and $0 < \alpha < 1$.

Results

As in [9, 10] an equivalent finite-time ISS Lyapunov function definition can be used instead of (15):

$$\|x\| \geq \chi(\|d\|) \to \sup_{\varphi \in F(x,d)} DV(x)\varphi \leq -\alpha_3(\|x\|)$$

for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^p$ with some $\alpha_3 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$.

Definition

A locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called finite-time iISS Lyapunov function for the system (12) if there are $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$ and a positive definite continuous function $\alpha_3 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^p$

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$

$$\sup_{\varphi \in F(x,d)} DV(x)\varphi \leq -\alpha_3(\|x\|) + \sigma(\|d\|),$$

with $\alpha_3(\|x\|) \succ aV^\alpha(x)$ for some $a > 0$ and $0 < \alpha < 1$.

Results

The definition of the finite-time ISS Lyapunov function is given for the global case, the local one can be obtained for $x \in \mathcal{V}$ and $d \in \mathcal{D}$. The main result of this subsection is as follows.

Theorem

[11] If for the system (12) there exists a finite-time ISS (finite-time iISS) Lyapunov function, then it is finite-time ISS (finite-time iISS).

Results

All definitions and the result of this subsection are also valid for ODEs (5), where the set $\mathbb{S}(x_0)$ for any $x_0 \in \mathbb{R}^n$ is composed by the unique solution $\Phi^t(x_0)$, and all formulations above have to be simplified accordingly.

Results

Finally we would like to describe the link between the robust stability and homogeneity. In the UFTS case, if the degree k of homogeneity of the DI (12) equals to $-\min_{1\leq i\leq n} r_i$, then there exists a set $\mathcal{D} \subset \mathcal{L}_\infty$ such that the system (12) is UFTS with respect to an additive input $d(t) \in \mathcal{D}$. For ODE (5) and ISS property, some results have been proposed in [**?**, **?**]. An extension of those results for (5) with inclusion of iISS property has been recently proposed by the authors [12]. Define $\tilde{f}(x,d) = [f(x,d)^T \; 0_p]^T \in \mathbb{R}^{n+p}$, it is an extended auxiliary vector field for the system (5).

Results

Theorem

[12] Let the vector field $\tilde{f}$ be homogeneous with the weights $\mathbf{r} = [r_1, \ldots, r_n] > 0$, $\tilde{\mathbf{r}} = [\tilde{r}_1, \ldots, \tilde{r}_p] \geq 0$ with a degree $k \geq -\min_{1 \leq i \leq n} r_i$, i.e. $f(\Lambda_r x, \Lambda_{\tilde{r}} d) = \lambda^k \Lambda_r f(x, d)$. Assume that the system (5) is globally asymptotically stable for $d = 0$, then the system (5) is

ISS *if $\tilde{r}_{\min} > 0$, where $\tilde{r}_{\min} = \min_{1 \leq j \leq p} \tilde{r}_j$,*

iISS *if $\tilde{r}_{\min} = 0$ and $k \leq 0$.*

Results

As we can conclude from this result, for homogeneous system (5) its robustness (ISS or iISS property) is a function of its degree of homogeneity. Thus to verify robustness of the system FTS with respect to an external input it is enough to compute its degree of homogeneity and perform some other algebraic operations, which is a big advantage of homogeneity. Interestingly to note, that FTS and iISS have a similar restriction on the degree of homogeneity: it has to be negative (non positive for iISS). For example, according to this theorem, if for some locally Lipschitz continuous and homogeneous f (with the degree k and the weights $\mathbf{r}$) we have $f(x,d) = f(x) + d$, i.e. d is an additive disturbance, then the system (5) is ISS if $k > -\min_{1\leq i\leq n} r_i$, and it is iISS for $k = -\min_{1\leq i\leq n} r_i$. If $f(x,d) = f(x+d)$ and d is a measurement noise, then the system is always ISS. An extension of this result for the DI (12) needs some additional conditions.

Results

Denote an extended discontinuous function
$\tilde{F}(x,d) = [F(x,d)^T \; 0_p]^T$.

Theorem

[11] Let the discontinuous function $\tilde{F}$ be homogeneous with the weights $\mathbf{r} = [r_1, \ldots, r_n] > 0$, $\tilde{\mathbf{r}} = [\tilde{r}_1, \ldots, \tilde{r}_p] \geq 0$ with a degree $k \geq -\min_{1\leq i\leq n} r_i$, i.e. $F(\Lambda_r x, \Lambda_{\tilde{r}} d) = \lambda^k \Lambda_r F(x,d)$. Assume that the system (12) is globally asymptotically stable for $d = 0$. Let also

$$\|F(y,d) - F(y,0)\|_H \leq \sigma(\|d\|), \quad \forall y \in S_r, \; \sigma(s) = \begin{cases} c\, s^{\varrho_{\min}} & \text{if } s \leq 1 \\ c\, s^{\varrho_{\max}} & \text{if } s > 1 \end{cases}$$

for some $c > 0$ and $\varrho_{\max} \geq \varrho_{\min} > 0$. Then the system (12) is

ISS *if $\tilde{r}_{\min} > 0$, where $\tilde{r}_{\min} = \min_{1\leq j\leq p} \tilde{r}_j$,*

iISS *if $\tilde{r}_{\max}\varrho_{\min} - \mu \leq \nu \leq \tilde{r}_{\min} = 0$, where $\tilde{r}_{\max} = \max_{1\leq j\leq p} \tilde{r}_j$.*

Results

Let us give the definition of the *finite time stabilization*.

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)u_i, \quad x \in \mathbb{R}^n \text{ and } u = (u_1, ..., u_m) \in \mathcal{U} \quad (16)$$

The control system (16) is *finite time stabilizable* if there is a feedback control law $u \in \mathcal{CL}^0(\mathcal{V}, \mathcal{U})$ such that:

1. $u(0) = 0$,
2. the origin is a finite time stable equilibrium of the closed-loop system:

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)u_i(x), \quad x \in \mathcal{X}. \quad (17)$$

Results

We recall some usual definitions. A positive definite function $V \in \mathcal{CL}^{\infty}(\mathcal{V}, \mathbb{R}_{\geq 0})$ is a *control Lyapunov function* for the system (16) if for all $x \in \mathcal{V} \setminus \{0\}$,

$$\inf_{u \in \mathcal{U}} \left(a(x) + \langle B(x), u \rangle \right) < 0.$$

where $a(x) = \langle \nabla V(x), f_0(x) \rangle$, $B(x) = (b_1(x), ..., b_m(x))$ with $b_i(x) = \langle \nabla V(x), f_i(x) \rangle$ for $1 \leq i \leq m$.

Results

To obtain finite time stabilization, one introduces for a control Lyapunov function V the following decrease condition which is for all $x \in \mathcal{V} \setminus \{0\}$,

$$\inf_{u \in \mathcal{U}} \left(a\left(x\right) + \left\langle B\left(x\right), u \right\rangle \right) \leq -c\left(V\left(x\right)\right)^{\alpha}. \tag{18}$$

Results

As usually, such a control Lyapunov function satisfies the *small control property* if for each $\epsilon > 0$, there is $\delta > 0$ such that, if $x \in \mathcal{V} \setminus \{0\}$, then there is some $u \in \epsilon \mathcal{B}^m$ such that:

$$a(x) + \langle B(x), u \rangle \leq -c(V(x))^{\alpha}.$$

Moreover, we set $b(x) = \|B(x)\|^2$.

Results

finite time stabilization:

Theorem

The system (16) is finite time stabilizable if and only if there is a control Lyapunov function for the system (16) which satisfies condition (18) and the small control property.

Results

The proof of theorem 14 is not constructive (Selection theorem). Let us give an explicit feedback for the system (16) with $\mathcal{U} = \mathbb{R}^m$ using the control given by Sontag in [6] adding a property to ensure the finite time stability of the closed-loop system. So, we need for a control Lyapunov function $V : \mathcal{V} \to \mathbb{R}_{\geq 0}$ the following condition:

$$a(x)^2 + b(x)^2 \geq s(V(x)) \tag{19}$$

for all $x \in \mathcal{V} \setminus \{0\}$, where $\sqrt{s}$ is a class $\mathcal{KI}$ function.

Results

Theorem

If there is a control Lyapunov function for the system (16) satisfying condition (19) with a class $\mathcal{KI}$ function $\sqrt{s}$, then the system (16) is almost finite time stabilizable by the Sontag feedback **??**. *Moreover, if the control Lyapunov function satisfies the small control property, then the system (16) is finite time stabilizable.*

Results

It seems to be difficult to find a control Lyapunov function satisfying condition (18) or condition (19) (more easy for homogeneous systems... amy look at homogeneous approximation).

Results

Example

Let us consider the system:

$$\dot{x} = x^5 + u, \quad x \in \mathbb{R},$$

and $\alpha, \beta \in]0,1[$ such that $\beta(\alpha+1) = 4\ \alpha$ (for example $\beta = \frac{1}{2}$ and $\alpha = \frac{1}{7}$). One defines the C^1-function $V(x) = |x|^{\alpha+1}$ which is proper, then:

$$\begin{aligned} a(x) &= (\alpha+1)\,|x|^{\alpha}\, sgn(x)\, x^5 \\ B(x) &= (\alpha+1)\,|x|^{\alpha}\, sgn(x) \\ b(x) &= (\alpha+1)^2\,|x|^{2\alpha}. \end{aligned}$$

Results

Example

As $\inf_{u\in\mathbb{R}}\left[a\left(x\right)+B\left(x\right)u\right]<0$ for $x\neq 0$, V is a control Lyapunov function for the system. Moreover, one obtains:

$$a\left(x\right)^{2}+b\left(x\right)^{2}\geq\left(\alpha+1\right)^{4}\left(V\left(x\right)\right)^{\beta}.$$

Results

Example

It is well known that $\int_0^{\epsilon} \frac{dz}{z^{\beta}} < +\infty$ for $\epsilon > 0$, so the proposition 15 guarantees that the system is globally finite time stabilizable with the control:

$$\boxed{u\left(x\right) = \frac{-|x|^{\alpha} sgn(x) x^{5} - \sqrt{|x|^{2\alpha} x^{10} + (\alpha+1)^{2} |x|^{4\alpha}}}{|x|^{\alpha} sgn(x)}.}$$

Results

An other way to obtain an explicit feedback with a control Lyapunov function for the system (16) satisfying condition (18) is to consider the function

$$u_i(x) = \begin{cases} -b_i(x) \dfrac{a(x) + c(V(x))^\alpha}{b(x)} & \text{if} \quad x \notin \mathcal{A} \\ a(x) & \text{if} \quad x \in \mathcal{A} \end{cases}$$

where $\mathcal{A} = \{x \in \mathcal{V} : b(x) = 0\}$, to study its continuity. If u is continuous, then it is a feedback for the system (16).

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