Homogeneous Evolution Equations

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Outline

Introduction

Different types of homogeneity

- **Standard homogeneity** (L. Euler in 17th century, ...):
  uniform dilation $x \to \lambda x$, where $\lambda > 0$ and $x \in \mathbb{R}^n$;

- **Weighted homogeneity** (Zubov 1958, Hermes 1986, Rosier 1992,...)
  non-uniform (anisotropic) dilation:
  $$(x_1, x_2, ..., x_n) \to (\lambda^{r_1} x_1, \lambda^{r_2} x_2, ..., \lambda^{r_n} x_n),$$
  where $\lambda > 0$, $x \in \mathbb{R}^n$, and $r_i > 0$, $i = 1, 2, ..., n$;

- **Geometric homogeneity** (Khomenyuk 1961, Kawski 1995, ...)
  generalized dilations of one vector field with respect to another one.

The goal

From Homogeneous ODE to Homogeneous PDE

Prehistory: Folland 1975, Benilan & Crandall 1981, Sanders & Wang 1998, ...

2. Homogenenous Evolution Equation

Model description

Let us consider the nonlinear evolution equation

$$\dot{u}(t) = f(u(t)), \quad t \in \mathbb{R}_+, \tag{1}$$

with the initial condition

$$u(0) = \varphi \in \Omega, \tag{2}$$

where

- $f : \Omega \subset \mathbb{B} \to \mathbb{B}$ is an operator with the domain Ω;
- $u(\cdot) \in \Omega \subset \mathbb{B}$ is the system state;
- B is a Banach Space.

We assume that Cauchy problem (1) - (2) has solutions in some (possibly generalized) sense.

2.1. Dilation in Banach Space

Dilation in Banach Space

Let $\mathcal{L}(\mathbb{B})$ be the space of linear bounded operators $\mathbb{B} \to \mathbb{B}$.

Definition

A map $\mathbf{d} : \mathbb{R} \to \mathcal{L}(\mathbb{B})$ is called **dilation** in the space $\mathbb{B}$ if it satisfies

- **the semigroup property:**
  $\mathbf{d}(0) = I \in \mathcal{L}(\mathbb{B})$ and $\mathbf{d}(t+s) = \mathbf{d}(t)\mathbf{d}(s)$ for $t, s \in \mathbb{R}$;
- **the continuity property:**
  the map $\mathbf{d}(\cdot)u : \mathbb{R} \to \mathbb{B}$ is continuous for any $u \in \mathbb{B}$;
- **the limit property:** (Terasaka's condition)
  $\lim\limits_{s\to-\infty} \|\mathbf{d}(s)u\| = 0$ and $\lim\limits_{s\to+\infty} \|\mathbf{d}(s)u\| = \infty$ uniformly on $u \in S$.

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The dilation $\mathbf{d}$ is a group

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- *Geometric dilation* : $\Omega = \mathbb{R}^n$, $\mathbf{d}$ is the flow of an Euler vector field.
- *Dilation in a Banach space*:

$$\Omega = \{u \in \mathbb{C}([0,p],\mathbb{R}) : u(0) = u^2(p)\}$$

and

$$(\mathbf{d}(s)u)(x) = e^{s-0.5sx/p}u(x), \text{ where } x \in [0,p].$$

$$(\mathbf{d}(s)u)(0) = e^s u(0) \text{ and } (\mathbf{d}(s)u)(p) = e^{0.5s}u(p) \quad \Rightarrow \quad \mathbf{d}(\cdot)\Omega = \Omega$$

2.2. Homogeneous Operator

Homogeneous Operator

Definition

An operator $f : \Omega \subset \mathbb{B} \to \mathbb{B}$ is said to be **d**-homogeneous of degree ν if its domain Ω is **d**-homogeneous set and

$$f(\mathbf{d}(s)u) = e^{\nu s}\mathbf{d}(s)f(u) \qquad s \in \mathbb{R}, \quad u \in \Omega, \tag{3}$$

where **d** is a dilation in $\mathbb{B}$ and $\nu \in \mathbb{R}$.

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Homogeneous Evolution Equation

The evolution equation (1) is said to be **d**-homogeneous if the corresponding operator $f : \Omega \subset \mathbb{B} \to \mathbb{B}$ is **d**-homogeneous.

Example 1: Korteweg-de Vries equation (KdV equation)

$$\frac{\partial v}{\partial t} = -\frac{\partial^3 v}{\partial x^3} - v\frac{\partial v}{\partial x},$$

where v is a scalar function of time $t \in \mathbb{R}_+$ and space $x \in \mathbb{R}$ variables.

Dilation

$(\mathbf{d}(s)u)(x) = e^{2s}u(e^s x)$, where $x \in \mathbb{R}$, $u \in \mathbb{C}^3(\mathbb{R}, \mathbb{R})$ and $s \in \mathbb{R}$

Example 2: Saint-Venant equation

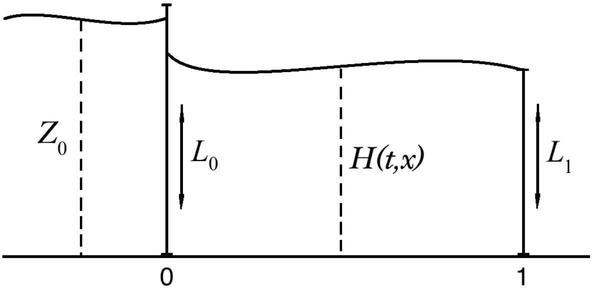

Figure : Water channel with two spillways

$$
\begin{aligned}
\frac{\partial H}{\partial t} &= -\frac{\partial}{\partial x}(HV), & H(t,0)V(t,0) - (Z_0 - L_0)^{3/2} = 0, \\
\frac{\partial V}{\partial t} &= -\frac{\partial}{\partial x}\left(\tfrac{1}{2}V^2 + gH\right), & H(t,1)V(t,1) - (H(t,1) - L_1)^{3/2} = 0.
\end{aligned}
$$

$H(t,x)$ - the water level and $V(t,x)$ is the water velocity.

Example 2: Saint-Venant equation (continuation)

Domain (for $Z_0 = L_0$ and $L_1 = 0$)

$$\Omega = \left\{ (u_1, u_2) \in \mathbb{C}^1([0,1], \mathbb{R}_+) \times \mathbb{C}^1([0,1], \mathbb{R}) : \begin{array}{c} u_1(0)u_2(0) = 0; \\ u_1(1)u_2(1) = u_1^{3/2}(1) \end{array} \right\}$$

Dilation

$$\mathbf{d}(s)u = (e^{2s}u_1, e^{s}u_2),$$

where $u = (u_1, u_2) \in \mathbb{C}([0,1], \mathbb{R}) \times \mathbb{C}([0,1], \mathbb{R})$ and $s \in \mathbb{R}$

2.3 Properties of Homogeneous Evolution

Properties of Homogeneous Evolution

$$\dot{x} = f(x), \; x(0) = x_0$$

	Homogeneous ODE $x \in \mathbb{R}^n$	**Homogeneous EE** $x \in \mathbb{B}$
Trajectory Scaling	$x(t, \Lambda x_0) = \Lambda x(\lambda^{\nu} t, x_0)$	$x(t, d(s)x_0) = d(s)x(e^{\nu s} t, x_0)$
Local $\Leftrightarrow$ Global	✓	✓
Invariance $\Leftrightarrow$ Stability	✓	**?**
Robustness (Input-to-State Stability)	$\dot{x} = f(x, w)$ $w \in \mathbb{L}^{\infty}$	**?**
Finite-time stability for negative degree	✓	✓

Example 3: Fast Diffusion Equation is homogeneous ($\nu < 0$)

$$\frac{\partial u}{\partial t} - \Delta\left(u^{1+\nu}\right) = 0, \quad \nu \in (-1, 0),$$

where Δ - the Laplace operator, u - a scalar function of $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$.

$$u(t, x) = 0 \text{ for } x \in \partial M,$$

where $M \in \mathbb{R}^n$ is a bounded connected domain with a smooth boundary.

Fast Diffusion Equation is $\mathbf{d}$-homogeneous($\mathbf{d}(s) = e^s$) with negative degree and **finite-time stable**.

Thank you for your attention