Finite-time stabilization of a system of conservation laws

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Finite-time stability

- A system $\dot{x} = f(t, x)$ is **finite-time** stable at the origin, if the origin is stable, and if all the trajectories reach the origin in **finite time**. Example:

  $$\dot{x} = -x^{\frac{1}{3}}$$

  Then $|x(t)| \leq |x_0|$ and $x(t) = 0$ for $t \geq \frac{3}{2}|x_0|^{\frac{2}{3}}$
- A **finite-time stabilizer** is a feedback control for which the closed-loop system is finite-time stable.
- It realizes a **control** objective with a control in **feedback form**, and it may be seen as an exponential stabilizer with an **arbitrary large** decay rate!

  $$|x(t)| \leq C(|x_0|)e^{-\lambda(t-T)}$$

Finite-time stabilizer for ODE's

- Non Lipschitz-continuous feedback control appear naturally in nonlinear automatic control. Solutions may fail to be unique, but they are often unique **forward in time**.
- Widely studied in the last decade, after Haimo '86 and Bhat-Berstein '98 (links with Lyapunov functions).
- Explicit finite-time stabilizer given for any linear cascade system (hence for any linear controllable system)
  Example: Double integrator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\mathrm{sign}(x_1)|x_1|^\alpha - \mathrm{sign}(x_2)|x_2|^\beta \end{aligned}$$

Finite-time stable if $0 < \alpha < 1$, and $\beta = \frac{2}{1+\alpha^{-1}}$ (Haimo '86)

What about PDE's?

- Finite-time extinction phenomenon (without control) exists for certain evolution equations.
- For the wave equation, the use of transparent boundary conditions leads to the finite-time stability (Majda 1975, Komornik 1994).

$$\begin{aligned} &y_{tt} - y_{xx} = 0, \qquad 0 < x < 1 \\ &y_{x|x=1} = -y_{t|x=1} \\ &y_{x|x=0} = y_{t|x=0} \\ &(y, y_t)_{|t=0} = (y_0, y_1) \end{aligned}$$

- **D'Alembert formula**:

$$y(t,x) = f(x-t) + g(x+t)$$

The wave $f(x-t)$ leaves the domain (0,1) at x=1, while the wave $g(x+t)$ leaves (0,1) at x=0

Riemann invariants

- With $u = y_x - y_t$, and $v = y_x + y_t$ (Riemann invariants)

$$u_t + u_x = 0, \quad u_{|x=0} = 0$$
$$v_t - v_x = 0, \quad v_{|x=1} = 0.$$

  so $u = v = 0$ (hence $y \equiv C$) for $t \geq 1$
- For $y_{x\,|x=1} = -y_{t\,|x=1}$ and $y_{|x=0} = 0$,

$$v_{|x=1} = 0, \quad u_{|x=0} = v_{|x=0}.$$

  so $u = v = 0$ (hence $y \equiv 0$) for $t \geq 2$

Systems of two conservation laws in diagonal form

- Similar result holds for a 2×2 hyperbolic system in diagonal form

$$
\begin{aligned}
&u_t + \lambda(u,v)u_x = 0\\
&v_t + \mu(u,v)v_x = 0\\
&u_{|x=0} = 0, \quad v_{|x=1} = 0
\end{aligned}
$$

  with

$$\mu(u,v) \leq -c < c \leq \lambda(u,v)$$

  some smooth functions and $c > 0$ some constant
- Noticed in **Leugering-Schmidt '02** for Saint-Venant system on an interval or on a star-shaped tree
- The initial conditions (u_0, v_0) have to fulfill the **compatibility conditions**

$$
\begin{aligned}
& u_0(0) = v_0(1) = 0, \quad \text{for a } C^0 - \text{solution}\\
\text{and also} \quad & u_0'(0) = v_0'(1) = 0, \quad \text{for a } C^1 - \text{solution}
\end{aligned}
$$

  Otherwise, **chocks appear** at $t = 0$ (**Gugat-Leugering '03**).

Dynamical boundary conditions

To avoid compatibility conditions, we introduce dynamical boundary conditions produced by a finite-time stable ODE:

$$\frac{d}{dt}u(t,0) = -K\text{sign}(u(t,0))|u(t,0)|^{\gamma} \quad (1)$$

$$\frac{d}{dt}v(t,1) = -K\text{sign}(v(t,1))|v(t,1)|^{\gamma} \quad (2)$$

where $(K,\gamma) \in (0,+\infty) \times (0,1)$ are some constants.
The system we consider is thus

$$u_t + \lambda(u,v)u_x = 0$$
$$v_t + \mu(u,v)v_x = 0$$
$$(u,v)_{|t=0} = (u_0,v_0)$$

supplemented with (1)-(2), where (u_0,v_0) can be **any** (Lipschitz-continuous) initial data. **Smooth solutions desirable**.

Complete system

$$u_t + \lambda(u,v)u_x = 0$$
$$v_t + \mu(u,v)v_x = 0$$

$$\dot{u}_l = -K\text{sign}(u_l)|u_l|^\gamma$$
$$\dot{v}_r = -K\text{sign}(v_r)|v_r|^\gamma$$

$$u(t,0) = u_l(t), \quad v(t,1) = v_r(t)$$
$$(u,v)_{|t=0} = (u_0,v_0), \quad (u_l,v_r)(0) = (u_0(0), v_0(1))$$

New state: $\big(u(t,.), v(t,.), u_l(t), v_r(t)\big) \in V$ where

$$V := \{(u,v,u_l,v_l) \in W^{1,\infty}(0,1)^2 \times \mathbb{R}^2 : \quad u_l = u(0),\ v_r = v(1)\}$$

Main results[1]

Let

$$T = \frac{1}{c} + \frac{C_1^{1-\gamma}}{(1-\gamma)K}$$
$$\max(||u_0||_\infty, ||v_0||_\infty) \le C_1$$
$$\max(||u_0'||_\infty, ||v_0'||_\infty) \le C_2$$

Thm 1. (Two controls) If C_1 and C_2 are small enough, then there is a unique solution (u, v) of the above system in the class

$$\mathcal{D} = \{(u, v) \in \text{Lip}\ ([0, T] \times [0, 1])^2;\ \max(||u||_\infty, ||v||_\infty) \le C_1\}$$

Furthermore,

- $u(T, .) = v(T, .) = 0$
- $||(u, v)||_{L^\infty(0,T;W^{1,\infty}(0,1))} \to 0$ as $||(u_0, v_0)||_{W^{1,\infty}(0,1)} \to 0.$

Thus we have **finite-time stability** in $\text{Lip}([0, 1]) = W^{1,\infty}(0, 1)$.

[1] V. Perrollaz, L. Rosier, *Finite-time stabilization of* 2×2 *hyperbolic systems on tree-shaped networks*, SIAM J. Control Optim., **52**, (2014), 143-163.

Comments

- Thm 1 can be applied to any hyperbolic system that can be put in diagonal form. This is the case for any system admitting Riemann invariants, in writing the hyperbolic system in terms of these new unknowns.
- Example 1. The $p-$system ($p \in C^1(\mathbb{R})$)

$$r_t - s_x = 0,$$
$$s_t - [p(r)]_x = 0$$

- Example 2. Shallow water equations

$$H_t + (HV)_x = 0,$$
$$V_t + (\frac{V^2}{2} + gH)_x = 0$$

- Example 3. Euler equations for barotropic compressible gaz

One control?

- Sometimes, only one control is available. The other boundary condition can be **imposed by the physical context**. For instance, we could have $u(t, 0) = h(t, v(t, 0))$, where h is smooth and $h(t, 0) = 0$ for $t \geq T_h$.
- A result similar to Thm 1 can be obtained, with a **larger** extinction time.
- To simplify, assume that $u(t, 0) = h(v(t, 0))$ with $h(0) = 0$, and still $v(t, 1) = v_r(t)$. Then

$$T = \frac{2}{c} + \frac{C_1^{1-\gamma}}{(1-\gamma)K}$$

Thm 2. (One control)

Let

$$T = \frac{2}{c} + \frac{C_1^{1-\gamma}}{(1-\gamma)K}$$
$$C_1' = \max(C_1, ||h||_{L^\infty(-C_1,C_1)})$$
$$\max(||u_0||_\infty, ||v_0||_\infty) \leq C_1,$$
$$\max(||u_0'||_\infty, ||v_0'||_\infty) \leq C_2$$

Thm 2. If C_1 and C_2 are small enough, then there is a unique solution (u, v) of the boundary initial-value problem in the class

$$\mathcal{D} = \{(u, v) \in \text{Lip}\,([0, T] \times [0, 1])^2; ||u||_\infty \leq C_1', ||v||_\infty \leq C_1\}$$

Furthermore,

- $u(T, .) = v(T, .) = 0$
- $||(u, v)||_{L^\infty(0,T;W^{1,\infty}(0,1))} \to 0$ as $||(u_0, v_0)||_{W^{1,\infty}(0,1)} \to 0.$

Thus we have **finite-time stability** in $\text{Lip}([0, 1]) = W^{1,\infty}(0, 1)$.

Application to the control of flow in a canal

- Consider first **one canal**. Canal: rectangular cross section, no slope, no friction
  A good model given by shallow water (or Saint-Venant) system

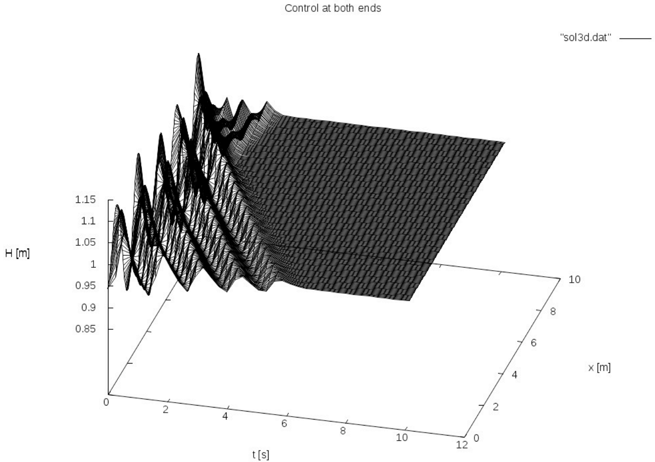
$$H_t + (HV)_x = 0,$$
$$V_t + (\frac{V^2}{2} + gH)_x = 0$$

- H = water depth, V = water velocity, and g = gravitation constant.
- Physically, input that can be controlled = flow rate

$$Q(t,x) = H(t,x)V(t,x)$$

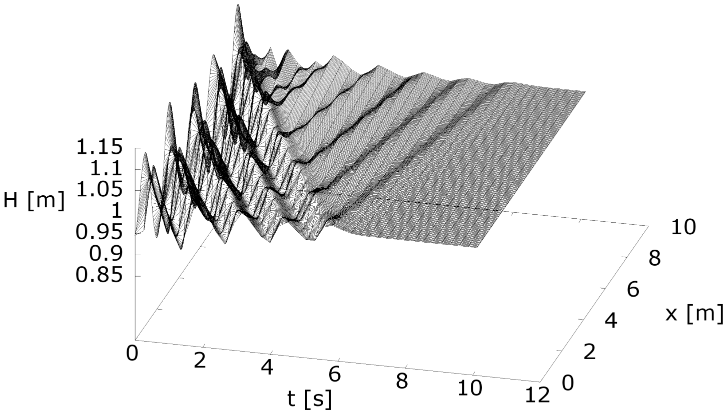
- Goal: stabilize the system around a constant equilibrium state (H^*, V^*). Set $Q^* = H^* V^*$.

Controls active at both $x = 0$ and $x = 1$: $T = 6.5$[2]

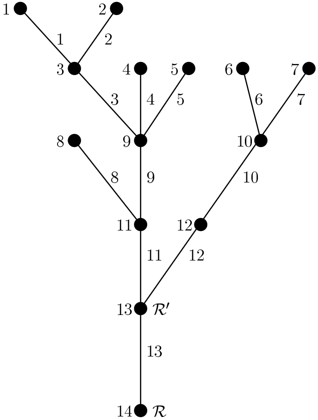


[2]V. Perrollaz, L. Rosier, *Finite-time stabilization of hyperbolic systems over a bounded interval*, in 1st IFAC workshop on Control of Systems Governed by Partial Differential Equations (CPDE2013), 2013, 239–244.

Control active at $x = 1$, no control at $x = 0$: $T = 9.2$



Network of canals



A tree with 14 nodes and depth equal to 5

Flow controls[3]

- At a simple node, we control the flow rate (hence u_i or v_i)
- At a multiple node, the **conservation of flows** (Kirchhoff law) implies that **one flow rate cannot be controlled** (being defined by the others). We control the incoming flows at the multiple node (hence the v_i), **not** the outcoming flow (hence some u_i is expressed as $u_i = h_i(t, v_i)$)
- We obtained by induction on the depth a **finite-time stabilization result with an extinction time** $T \sim d/c$, where $d =$depth (assuming all the edges of length one).

[3] V. Perrollaz, L. Rosier, *Finite-time stabilization of* 2×2 *hyperbolic systems on tree-shaped networks*, SIAM J. Control Optim., **52**, (2014), 143-163.

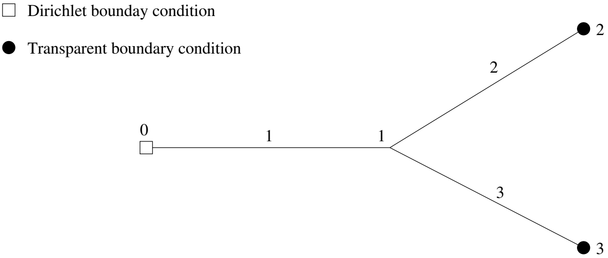
Finite-time stability for the wave equation en a tree

Joint work with **Fatiha Alabau** and **Vincent Perrollaz**[4]

- "Toy" problem to understand what happens when less controls are available
- At each external node (but one): transparent boundary condition (Dirichlet b.c. at the root)
- At each internal node connecting k edges: $k-1$ continuity conditions + Kirchhoff law with a damping term

[4] F. Alabau-Boussouira, V. Perrollaz, L. Rosier, *Finite-time stability of a network of strings*, MCRF, Vol. 5, No. 4, Dec. 2015.

Example: star-shaped tree with 3 edges



$$
\begin{aligned}
&u_{i,tt} - c_i^2 u_{i,xx} = 0, \qquad 0 < t < T,\ 0 < x < 1,\ i = 1,2,3 \\
&c_i u_{i,x}(t,1) = -u_{i,t}(t,1), \qquad i = 2,3 \\
&u_1(t,0) = 0, \\
&u_1(t,1) = u_2(t,0) = u_3(t,0) \\
&c_2 u_{2,x}(t,0) + c_3 u_{3,x}(t,0) - c_1 u_{1,x}(t,1) = -\alpha u_{1,t}(t,1)
\end{aligned}
$$

The results

Pick any tree, and apply at all the external nodes (but one) transparent b.c., and Dirichlet b.c. at the root.

- **Thm 1 (Well-posedness)** The well-posedness in the energy space occurs iff

$$\alpha(n) \neq k$$

at each internal node n connecting k edges

- **Thm 2 (Finite-time stability)** If

$$\alpha(n) = k - 2 \qquad (*)$$

at each internal node n connecting k edges, we have finite time stability in the energy space.
Extinction time = maximum of the quantities

$$2(c_{i_1}^{-1} + c_{i_2}^{-1} + \cdots + c_{i_p}^{-1})$$

where the edges of indices i_1, ..., i_p form a path from the root to an external node

- **Thm 3** Condition $(*)$ is **sharp** for a **star-shaped tree**

With transparent b.c. at each external node

- The finite-time stability still occurs under

$$\alpha(n) = k - 2 \qquad (*)$$

for each internal node n

Extinction time = maximum of the quantities

$$c_{i_1}^{-1} + c_{i_2}^{-1} + \cdots + c_{i_p}^{-1}$$

where the edges of indices i_1, ..., i_p form a path from one external node to another one

- However, $(*)$ is not sharp!
  1. For a star-shaped tree, the finite time stability holds for **any** $\alpha \neq k$ (finite time stab. noticed by Gugat-Sigalotti (2010) when $\alpha = 0$)
  2. For a bone-shaped tree (2 internal nodes), the finite time stability holds iff $(*)$ holds for **at least** one internal node.

Conclusion

- We provided boundary feedback laws leading to a finite-time stability with 2 (or 1) controller(s), using only local measurements
- Extension to any tree-shaped network
- Questions of interest:
  - effective regularity of the solutions;
  - semi-global stability;
  - addition of source terms $\varepsilon f(u, v)$ (friction, slope) in the diagonal system; decay of the L^∞ norm (probably) like $Ce^{-Ct\log(\varepsilon^{-1})}$ (work in progress with M. Gugat and V. Perrollaz)
  - robustness with respect to shocks at $t = 0$
  - Sharpness of the choice of the damping coefficients at the internal nodes for any tree of strings; nonlinear counterpart?
  - cycles?
  - numerical control of the semilinear wave equation (work in progress with Marcelo & Valeria Cavalcanti, Gilles Lebeau, Carole Rosier)